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Physically motivated invariant formulation for transversely isotropic hyperelasticity

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Abstract

This article discusses an invariant formulation for transversely isotropic hyperelasticity. The work is motivated by the interest of modeling materials such as tendon tissues which may exhibit drastically different characteristics in tensile, shear and volumetric responses. A multiplicative decomposition of the deformation gradient that factors out the dilation and the fiber stretch is proposed. Transversely isotropic strain invariants are constructed on the basis of the multiplicative factors. Within the framework of hyperelasticity theory, these strain invariants generate decoupled stress components in the hydrostatic pressure, the fiber tension and shear terms. An example model is suggested and is assessed against some known features of transversely isotropic solids with strong fibers.

Keywords: Transversely isotropic hyperelasticity; Anisotropic invariants; Stress and strain decompositions

1. Introduction

This work is motivated by the interest of modeling finitely deforming transversely isotropic elastic solids that exhibit strong anisotropies in their stress response. Materials of interest include composites reinforced with one family of strong fibers, and biological materials such as tendon or ligament tissues. The tissues in a tendon consist of parallelly structured collagen fibers. The ligament tissue has a similar structure, but the fibers can be less regularly aligned (Fung, 1993). In these tissues, the tensile response in the fiber direction depends primarily on the tensile property of the collagen fibers, which are relatively strong and stiffen significantly when subject to large stretch. On the other hand, the shear stress between fibers is determined

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mainly by the property of the ground substance that bonds the fibers and by fiber-matrix interaction. Due to the different mechanical characteristics of the fibers and the ground substance, the material can have distinct behaviors in tension and shear motion. Within the context of hyperelastic theory, the macroscopic description of the material response is given in terms of the strain energy functions which depend on certain strain invariants. To effectively reflect the microstructural composition in the constitutive description, it is desirable to use strain invariants that can register some kinematic modes for which the mechanical responses are distinct.

In a series of papers (Criscione et al., 2001, 2002; Criscione and Hunter, 2003), Criscione and coworkers have derived sets of strain invariants that represent succinct kinematic modes for the deformation of elastic material with one- or two-families of fibers. In Criscione et al. (2001), a set of five strain invariants is derived for transversely isotropic solids. These invariants register some distinct kinematic modes including the dilation, the fiber stretch and two shear modes. Within the framework of hyperelasticity, these invariants yield five stress terms in the hydrostatic pressure, the fiber tension and two shear terms, which are almost mutually orthogonal. The orthogonality between stress components offers a unique advantage in the experimental determination of the energy functions. Since the invariant approach in Criscione et al. (2001) leads to separated stress response in the pressure, the fiber tension and the shear stresses, it can be directly used to facilitate the physically motivated modeling interested in this work. However, these invariants, derived from rigorous kinematic analysis, involve transcendental functions and may not be convenient in analysis. In this work, we propose an alternative approach for constructing these physically based strain invariants. The construction relies on a simple kinematic decomposition that factors out the dilation and the fiber stretch only. The strain invariants so obtained also generate distinct stress components that carry clear physically meanings for transversely isotropic solids.

The present approach is a logical extension of the isochoric/volumetric split used in the analysis of isotropic solids. The decomposition was initially proposed by Flory (1961) in the analysis of rubber elasticity, see also Ogden (1984). Physically, it is motivated by the premise that the dilation and the deviatoric responses of rubber-like materials are sustained by different mechanisms. Using separated volume and isochoric strains in a hyperelasticity energy function, the ensuing pressure and deviatoric stresses are automatically decoupled, and thus allowing them to be characterized separately. Today, this formulation has been widely used in the modeling and finite element simulation of nearly incompressible hyperelastic materials (Simo et al., 1985; Simo and Taylor, 1991), and in particular biological tissues (Weiss et al., 1996; Holzapfel et al., 2000). In this work, we also propose a stress computation procedure that carries over the essential computational structure developed by Simo et al. (1985) for the isochoric/volumetric decomposition.

The paper is organized as follows. Section 2 contains a brief background on the mathematical representation of transversely isotropic functions. In Section 3, we introduce a decomposition of the Cauchy stress meaningful to transversely isotropic solids. A multiplicative split of the deformation gradient is discussed in Section 4. Invariant formulations derived from the multiplicative factors are presented in Section 5, followed by the introduction of a computational procedure for the Cauchy stress in the proposed invariant formulation. In Section 7, we provide some assessment for an example model, the behavior of which is evaluated against some known features of the deformation in materials with strong fibers.

2. Continuum mechanics foundations

A material is transversely isotropic if its properties are indistinguishable in all the directions transverse to a preferred direction $\mathbf{N}$ in the undistorted reference configuration. For a hyperelastic solid characterized by a strain energy function of the deformation gradient $\mathbf{F}$, transverse isotropy requires that the energy function be invariant under the symmetry transformation $\mathbf{F} \to \mathbf{F}\mathbf{G}^{\mathrm{T}}$, where $\mathbf{G}$ is any member of the transversely isotropic symmetry group $\mathcal{G} = \{\mathbf{G} \in Orth(3) : \ \mathbf{GN} = \pm\mathbf{N}\}$, here $Orth(3)$ stands for the orthogonal group. The requirement of invariance under superposed rigid body motions renders the strain energy a function of the Green-Cauchy deformation tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$. In terms of $W = W(\mathbf{C})$, transverse isotropy requires

$$W(\mathbf{C}) = W(\mathbf{GCG}^{\mathrm{T}}) \quad \forall \mathbf{G} \in \mathcal{G}, \tag{1}$$

which indicates that W is a transversely isotropic scalar function of $\mathbf{C}$. It is known that such a function can be generated using the following strain invariants (Spencer, 1982)

$$\{I_1,\ I_2,\ I_3,\ I_4,\ I_5\}, \tag{2}$$

where

$$I_1 = \operatorname{tr}\mathbf{C}, \quad I_2 = \frac{1}{2}(I_1^2 - \operatorname{tr}\mathbf{C}^2), \quad I_3 = \det\mathbf{C}, \quad I_4 = \mathbf{C}\cdot\mathbf{N}\otimes\mathbf{N}, \quad I_5 = \mathbf{C}^2\cdot\mathbf{N}\otimes\mathbf{N}.$$

The notation $(\cdot)$ stands for the inner product between two tensors (including two vectors), and $\otimes$ means the tensor product defined by $(\mathbf{p}\otimes\mathbf{q})\mathbf{r} = \mathbf{p}(\mathbf{q}\cdot\mathbf{r})$. In (2), I_1, I_2, I_3 are the isotropic principal invariants of $\mathbf{C}$, while I_4, I_5 are two *transversely isotropic* strain invariants. In general, any five invariants that are in one-to-one correspondence with (2) can serve as the basis for the energy function. The particular set (2) has been widely used in analytical works, see Merodio and Ogden (2005) and the references therein, and is also preferred in numerical simulations since they are easy to compute. Geometrically, I_1 and I_2 register the average stretches of all line and the area elements at a material point. The invariant I_4 is the square stretch of the line element tangent to the fiber direction $\mathbf{N}$. As argued in Merodio and Ogden (2003), if the coordinate axis X_1 is taken to be in the fiber direction $\mathbf{N}$, then $I_5 = I_4 + C_{12}^2 + C_{13}^2$, indicating that I_5 registers the shear strains C_{12} and C_{13}. Recently, Schröder and Neff (2003) studied the polyconvexity conditions of the strain energy function in connection with the invariant set

$$\{I_4, \quad K_1 = I_1 - I_4, \quad K_2 = \mathbf{C}^*\cdot\mathbf{N}\otimes\mathbf{N}, \quad K_3 = I_2 - K_2, \quad I_3\}, \tag{3}$$

where $\mathbf{C}^* = I_3\mathbf{C}^{-1}$ is the adjugate of $\mathbf{C}$. By Nanson's formula, the area element with normal $\mathbf{N}$ in the reference configuration transforms as $(\det\mathbf{F})\mathbf{F}^{-\mathrm{T}}\mathbf{N}$. It follows that K_2 is the square stretch of the area element with normal $\mathbf{N}$ in the reference configuration. Hence, I_4 and K_2 single out the line and area changes in the fiber direction. The set $\{\operatorname{tr}\mathbf{U}, \operatorname{tr}\mathbf{U}^*, \det\mathbf{U}, |\mathbf{UN}|, |\mathbf{U}^*\mathbf{N}|\}$, where $\mathbf{U} = \sqrt{\mathbf{C}}$ and $\mathbf{U}^*$ is its adjugate, was used in a recent work by Steigmann (2003).

A rational method for constructing anisotropic functions with certain symmetry group was developed by Boehler (1979), see also Zheng (1994) for a survey of recent developments on this subject. Boehler proved that an anisotropic function with a certain symmetry group can be represented as an isotropic functions with the inclusion of suitable *structural tensors* in the argument list. It has been established (Boehler, 1979; Zheng and Spencer, 1993) that transverse isotropy can be characterized by a single structural tensor $\mathbf{A} = \mathbf{N}\otimes\mathbf{N}$. Therefore, a transversely isotropic strain energy in terms of the deformation tensor $\mathbf{C}$ can be expressed as an isotropic function in $\mathbf{C}$ and $\mathbf{A}$, namely

$$W(\mathbf{C}) = \hat{W}(\mathbf{C},\mathbf{A}) : \ \hat{W}(\mathbf{C},\mathbf{A}) = \hat{W}(\mathbf{GCG}^{\mathrm{T}}, \mathbf{GAG}^{\mathrm{T}}) \quad \forall \mathbf{G} \in Orth(3). \tag{4}$$

Using the classical isotropic representation theorems (Smith, 1971), it can be readily concluded that such a function can be generated by the strain invariants (2) or their equivalents. For details, see Boehler (1979) and Zheng and Spencer (1993). Similarly, a symmetric tensor-valued anisotropic function of $\mathbf{C}$ can be expressed as an isotropic tensor-valued function of $\mathbf{C}$ and $\mathbf{A}$. Consequently, such a function can be generated from the following tensorial basis (Boehler, 1979; Zheng, 1994):

$$\{\mathbf{I},\ \mathbf{A},\ \mathbf{C},\ \mathbf{C}^2,\ \mathbf{CA}+\mathbf{AC},\ \mathbf{C}^2\mathbf{A}+\mathbf{AC}^2\ \}. \tag{5}$$

Any nontrivial invariants in the set (2) generated from a transversely isotropic *tensor* function of **C** are transversely isotropic *scalar* functions of **C**. On the other hand, the derivative of a transversely isotropic *scalar* function relative to **C** produces a transversely isotropic *tensor* function of **C**. These facts will be exploited in the ensuing development.

3. An additive split of the Cauchy stress

In this section we propose a stress decomposition meaningful for transversely isotropic solids. It will be shown that, given the current fiber direction, the Cauchy stress can be uniquely decomposed into the sum of the hydrostatic pressure, the fiber tension and two shear terms. The decomposition is an extension of the additive split of stress introduced by Spencer (1992) in the context of formulating yield criterion for transversely isotropic materials. Spencer introduced a stress component

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \alpha_1 \mathbf{1} - \alpha_2 \mathbf{n} \otimes \mathbf{n}, \tag{6}$$

where $\mathbf{1}$ is the second order identity tensor, and $\mathbf{n}$ is the fiber direction in the current configuration. (In Spencer (1992), $\mathbf{n}$ was taken to coincide with $\mathbf{N}$ since small strain was concerned.) The component $\tilde{\boldsymbol{\sigma}}$ is required to be deviatoric and tension free in the fiber direction, that is

$$\mathrm{tr}\,\tilde{\boldsymbol{\sigma}} = 0, \quad \mathbf{n} \cdot \tilde{\boldsymbol{\sigma}}\mathbf{n} = 0. \tag{7}$$

These two conditions give

$$\alpha_1 = \frac{1}{2}(\mathrm{tr}\,\boldsymbol{\sigma} - \sigma_n), \quad \alpha_2 = \frac{1}{2}(3\sigma_n - \mathrm{tr}\,\boldsymbol{\sigma}), \tag{8}$$

where $\sigma_n = \boldsymbol{\sigma} \cdot \mathbf{n} \otimes \mathbf{n}$. Introducing the notations

$$\mathbf{a} = \mathbf{n} \otimes \mathbf{n}, \quad \bar{\mathbf{a}} = \mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{1}, \quad \boldsymbol{\sigma}_1 = \frac{1}{3}(\mathrm{tr}\,\boldsymbol{\sigma})\mathbf{1}, \quad \boldsymbol{\sigma}_2 = \frac{3}{2}(\boldsymbol{\sigma} \cdot \bar{\mathbf{a}})\bar{\mathbf{a}}, \tag{9}$$

the decomposition (6) can be written as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \tilde{\boldsymbol{\sigma}}. \tag{10}$$

Physically, $\boldsymbol{\sigma}_1$ is the hydrostatic pressure, $\boldsymbol{\sigma}_2$ is the deviatoric tension stress in the fiber direction. In this work, $\tilde{\boldsymbol{\sigma}}$ is further decomposed into two distinct shear terms. To this end, introduce

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_4, \tag{11}$$

where

$$\boldsymbol{\sigma}_3 = \mathbf{a}\tilde{\boldsymbol{\sigma}} + \tilde{\boldsymbol{\sigma}}\mathbf{a}, \quad \boldsymbol{\sigma}_4 = \tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_3. \tag{12}$$

Invoking (6), $\boldsymbol{\sigma}_3$ can be expressed as

$$\boldsymbol{\sigma}_3 = \mathbf{a}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{a} - 2(\boldsymbol{\sigma} \cdot \mathbf{a})\mathbf{a}. \tag{13}$$

A direct computation shows that

$$\boldsymbol{\sigma}_3\mathbf{n} = \boldsymbol{\sigma}\mathbf{n} - \sigma_n\mathbf{n}. \tag{14}$$

This equation indicates that $\boldsymbol{\sigma}_3$ is the shear stress cross the transverse plane, referred to as the transverse shear hereafter. By the symmetry of the Cauchy stress, it also equals the shear stress acting along the fiber between adjacent fibers. On the other hand, $\boldsymbol{\sigma}_4$ satisfies

$$\boldsymbol{\sigma}_4\mathbf{n} = \mathbf{0}, \tag{15}$$

which implies that $\boldsymbol{\sigma}_4$ is a plane stress in the transverse plane, namely, only the in-plane components of $\boldsymbol{\sigma}_4$ are nonzero. Further, since $\boldsymbol{\sigma}_4$ is deviatoric and tension-free in the fiber direction, it satisfies the equation

$$\boldsymbol{\sigma}_4 \cdot (\mathbf{1} - \mathbf{n} \otimes \mathbf{n}) = 0, \tag{16}$$

which implies that $\boldsymbol{\sigma}_4$ represents a pure shear stress.

If the coordinates in the current configuration are chosen such that the axis x_1 aligns with $\mathbf{n}$, the stress invariants defined above take the forms

$$[\boldsymbol{\sigma}_1] = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad [\boldsymbol{\sigma}_2] = \frac{2\sigma_{11} - \sigma_{22} - \sigma_{33}}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

and

$$[\boldsymbol{\sigma}_3] = \begin{bmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & 0 & 0 \\ \sigma_{31} & 0 & 0 \end{bmatrix}, \qquad [\boldsymbol{\sigma}_4] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}(\sigma_{22} - \sigma_{33}) & \sigma_{23} \\ 0 & \sigma_{32} & \frac{1}{2}(\sigma_{33} - \sigma_{22}) \end{bmatrix}.$$

The fact that $\boldsymbol{\sigma}_4$ is a pure shear stress can be readily seen from the component form. Since $(\sigma_4)_{22} + (\sigma_4)_{33} = 0$, we can annihilate $(\sigma_4)_{22}$ and $(\sigma_4)_{33}$ simultaneously by properly rotating the coordinate axes around $\mathbf{n}$, resulting in a pure shear stress.

The proposed stress decomposition is facilitated by (10) and (11). The uniqueness of the decomposition follows by construction. An important property of this decomposition lies in that the stress terms are mutually orthogonal. To show the orthogonality, we write the decomposition in terms of stress projections:

$$\boldsymbol{\sigma}_i = \mathbb{P}_i \boldsymbol{\sigma}, \quad i = 1, 2, 3, 4, \tag{17}$$

where the operation is defined in component by $(\sigma_i)_{pq} = [\mathbb{P}_i]_{pqst} \sigma_{st}$, and

$$\begin{aligned} \mathbb{P}_1 &= \frac{1}{3} \mathbf{1} \otimes \mathbf{1}, \\ \mathbb{P}_2 &= \frac{3}{2} \bar{\mathbf{a}} \otimes \bar{\mathbf{a}}, \\ \mathbb{P}_3 &= \mathbf{1} \boxtimes \mathbf{a} + \mathbf{a} \boxtimes \mathbf{1} - 2\mathbf{a} \otimes \mathbf{a}, \\ \mathbb{P}_4 &= \mathbb{I} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1} - \frac{3}{2} \bar{\mathbf{a}} \otimes \bar{\mathbf{a}} + 2\mathbf{a} \otimes \mathbf{a} - \mathbf{1} \boxtimes \mathbf{a} - \mathbf{a} \boxtimes \mathbf{1}. \end{aligned} \tag{18}$$

Here, $\mathbb{I}$ is the fourth order identity tensor, and $\boxtimes$ stands for the *Kronecker product* of second order tensors defined by

$$(\mathbf{U} \boxtimes \mathbf{V})(\mathbf{u} \otimes \mathbf{v}) = (\mathbf{U}\mathbf{u}) \otimes (\mathbf{V}\mathbf{v}) \quad \forall \text{ vectors } \mathbf{u}, \mathbf{v}. \tag{19}$$

A complete account of the properties of the Kronecker product can be found in the classical monograph of Murnaghan (1938). Here, we record only the multiplication rule $(\mathbf{U} \boxtimes \mathbf{V})(\mathbf{X} \boxtimes \mathbf{Y}) = (\mathbf{U}\mathbf{X}) \boxtimes (\mathbf{V}\mathbf{Y})$, which is needed in verifying the orthogonality condition stated below. Using this rule and the fact that $\operatorname{tr} \mathbf{a} = \mathbf{n} \cdot \mathbf{n} = 1$, one can readily check that

$$\mathbb{P}_i \mathbb{P}_i = \mathbb{P}_i \ (i = 1, 2, 3, 4), \quad \mathbb{P}_i \mathbb{P}_j = \mathbb{O} \ (i \neq j), \tag{20}$$

where $\mathbb{O}$ is the fourth-order zero tensor. Hence, $\mathbb{P}_i$ are identified as *orthogonal projectors*. It follows that the terms $\boldsymbol{\sigma}_1$ through $\boldsymbol{\sigma}_4$ are mutually orthogonal, as

$$\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = [\mathbb{P}_i \boldsymbol{\sigma}] \cdot [\mathbb{P}_j \boldsymbol{\sigma}] = \boldsymbol{\sigma} \cdot \mathbb{P}_i \mathbb{P}_j \boldsymbol{\sigma} = 0 \quad (i \neq j), \tag{21}$$

where the symmetry $[\mathbb{P}_i]_{pqst} = [\mathbb{P}_i]_{stpq}$ is used. In addition, it is evident that

$$\mathbb{P}_1 + \mathbb{P}_2 + \mathbb{P}_3 + \mathbb{P}_4 = \mathbb{I}. \tag{22}$$

Therefore, by virtue of these projections, the stress space $\mathscr{S}$ is decomposed into the *direct sum* of four subspaces, as

$$\mathscr{S} = \mathscr{S}_1 \oplus \mathscr{S}_2 \oplus \mathscr{S}_3 \oplus \mathscr{S}_4 \quad \text{where } \mathscr{S}_i = \{\boldsymbol{\tau} : \mathbb{P}_i\boldsymbol{\tau} = \boldsymbol{\tau}\}, \quad i = 1, 2, 3, 4. \tag{23}$$

Physically, $\mathscr{S}_1$ through $\mathscr{S}_4$ correspond to the spaces of the hydrostatic pressure, the deviatoric fiber tension, the transverse shear and the in-plane shear, respectively.

4. A multiplicative split of the deformation gradient

The construction of strain measures starts with the introduction of a multiplicative decomposition of the deformation gradient that factors out the volumetric strain and the fiber stretch. The split is an extension of the isochoric/volumetric decomposition widely used in the analysis of isotropic hyperelastic solids (Flory, 1961; Ogden, 1984; Simo et al., 1985; Simo and Taylor, 1991). It will be shown that, within the framework of hyperelasticity, the multiplicative factors correspond naturally (in sense of work-conjugancy) to the stress decomposition introduced in the previous section.

Let $\mathbf{n}$ be the fiber direction in the current configuration, given by the standard formula

$$\mathbf{n} = \frac{1}{\lambda}\mathbf{FN}, \quad \lambda^2 = \mathbf{FN}\cdot\mathbf{FN}, \tag{24}$$

where λ is the stretch of the line element along the fiber direction $\mathbf{N}$, $\lambda = \sqrt{I_4}$. Consider a decomposition of the deformation gradient in the form

$$\mathbf{F} = J^{\frac{1}{3}}(\alpha\mathbf{1} + \beta\mathbf{n}\otimes\mathbf{n})\tilde{\mathbf{F}}, \tag{25}$$

where $J = \det \mathbf{F} = \sqrt{I_3}$. From (25),

$$\tilde{\mathbf{F}} = J^{-\frac{1}{3}}(\alpha\mathbf{1} + \beta\mathbf{n}\otimes\mathbf{n})^{-1}\mathbf{F}. \tag{26}$$

We require that $\tilde{\mathbf{F}}$ to be isochoric and stretch-free in the fiber direction, namely

$$\det \tilde{\mathbf{F}} = 1, \quad \|\tilde{\mathbf{F}}\mathbf{N}\| = 1. \tag{27}$$

Making use of Eq. (24) and the relations

$$\begin{aligned}
&(\alpha\mathbf{1} + \beta\mathbf{n}\otimes\mathbf{n})^{-1} = \alpha^{-1}(\mathbf{1} - \mathbf{n}\otimes\mathbf{n}) + (\alpha+\beta)^{-1}\mathbf{n}\otimes\mathbf{n},\\
&\det(\alpha\mathbf{1} + \beta\mathbf{n}\otimes\mathbf{n}) = \alpha^2(\alpha+\beta),
\end{aligned} \tag{28}$$

we can write the conditions (27) as

$$\alpha^2(\alpha+\beta) = 1, \quad \alpha+\beta = \bar{\lambda}, \tag{29}$$

where $\bar{\lambda} = J^{-\frac{1}{3}}\lambda$, which is an isochoric measure of the fiber stretch. It can be readily found that

$$\alpha = \bar{\lambda}^{-\frac{1}{2}}, \quad \beta = \bar{\lambda} - \bar{\lambda}^{-\frac{1}{2}}. \tag{30}$$

Therefore,

$$\tilde{\mathbf{F}} = \left[\bar{\lambda}^{\frac{1}{2}}(\mathbf{1} - \mathbf{n}\otimes\mathbf{n}) + \bar{\lambda}^{-1}\mathbf{n}\otimes\mathbf{n}\right]J^{-\frac{1}{3}}\mathbf{F}. \tag{31}$$

The factor $\tilde{\mathbf{F}}$ represents a local motion composed of an isochoric deformation superposed by a simple compression in the fiber direction such that the ensuing fiber stretch is unity. Further insight into the nature of $\tilde{\mathbf{F}}$ can be gained by examining its rate. Invoking the standard results

$$\dot{\lambda} = \frac{\partial \lambda}{\partial \mathbf{C}} \cdot \dot{\mathbf{C}} = \lambda \mathbf{a} \cdot \mathbf{d}, \quad \dot{J} = J \,\mathrm{tr}\, \mathbf{d}, \quad \dot{\mathbf{n}} = \mathbf{L}\mathbf{n} - (\mathbf{d} \cdot \mathbf{a})\mathbf{n}, \tag{32}$$

where $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ is the velocity gradient and $\mathbf{d} = 1/2(\mathbf{L} + \mathbf{L}^{\mathrm{T}})$ is the rate of deformation tensor, a straight forward derivation yields

$$\tilde{\mathbf{L}} \equiv \dot{\tilde{\mathbf{F}}}\tilde{\mathbf{F}}^{-1} = \mathbf{L} - \frac{1}{3}(\mathbf{d} \cdot \mathbf{1})\mathbf{1} - \frac{3}{2}(\mathbf{d} \cdot \bar{\mathbf{a}})\bar{\mathbf{a}} + 2(\bar{\lambda}^{-\frac{3}{2}} - 1)[\mathbf{a}\mathbf{d} - (\mathbf{d} \cdot \mathbf{a})\mathbf{a}]. \tag{33}$$

Hence,

$$\tilde{\mathbf{d}} \equiv \frac{1}{2}(\tilde{\mathbf{L}} + \tilde{\mathbf{L}}^{\mathrm{T}}) = \left[\mathbb{P}_4 + \bar{\lambda}^{-\frac{3}{2}}\mathbb{P}_3\right]\mathbf{d} = \mathbb{P}\mathbf{d}, \quad \text{where } \mathbb{P} \equiv \mathbb{P}_4 + \bar{\lambda}^{-\frac{3}{2}}\mathbb{P}_3. \tag{34}$$

Recalling the orthogonality of the projectors, one sees

$$\tilde{\mathbf{d}} \cdot \mathbf{1} = 0, \quad \tilde{\mathbf{d}} \cdot \mathbf{a} = 0, \tag{35}$$

namely, the spatial rate of $\tilde{\mathbf{F}}$ is deviatoric and stretch-free in the fiber direction. As will become evident shortly afterwards, these conditions imply that the Cauchy stress generated by $\tilde{\mathbf{F}}$ is work-conjugate to the transverse shear and the in-plane shear stress.

5. Constitutive formulation

5.1. Strain invariants

Transversely isotropic strain invariants are constructed with the aid of the multiplicative split introduced in the previous section. Since J and $\bar{\lambda}$ can be naturally identified as two strain invariants, it makes sense to start with the energy function of the form $W = \breve{W}(J, \bar{\lambda}, \tilde{\mathbf{F}})$. Under superposed rigid motion $\mathbf{F} \to \mathbf{Q}\mathbf{F}$, J and $\bar{\lambda}$ remain invariant while $\tilde{\mathbf{F}}$ transforms as $\tilde{\mathbf{F}} \to \mathbf{Q}\tilde{\mathbf{F}}$. The classical requirement of invariance under superposed rigid body motions renders the reduction

$$W = \tilde{W}(J, \bar{\lambda}, \tilde{\mathbf{C}}), \tag{36}$$

where $\tilde{\mathbf{C}} = \tilde{\mathbf{F}}^{\mathrm{T}}\tilde{\mathbf{F}}$. From (31),

$$\tilde{\mathbf{C}} = J^{-\frac{2}{3}}\left[\bar{\lambda}\mathbf{C} + (\bar{\lambda}^{-2} - \bar{\lambda})\frac{1}{\lambda^2}\mathbf{C}\mathbf{N} \otimes \mathbf{N}\mathbf{C}\right]. \tag{37}$$

Evidently, $\tilde{\mathbf{C}}$ is a transversely isotropic *tensor function* of $\mathbf{C}$. Therefore, any invariants in the set (2) generated by $\tilde{\mathbf{C}}$ are transversely isotropic *scalar functions* of $\mathbf{C}$. However, since

$$I_3(\tilde{\mathbf{C}}) = \det \tilde{\mathbf{C}} = (\det \tilde{\mathbf{F}})^2 = 1, \quad I_4(\tilde{\mathbf{C}}) = \tilde{\mathbf{C}} \cdot \mathbf{N} \otimes \mathbf{N} = \tilde{\mathbf{F}}\mathbf{N} \cdot \tilde{\mathbf{F}}\mathbf{N} = 1 \tag{38}$$

only the following three invariants of $\tilde{\mathbf{C}}$ are nontrivial and are suitable for basis functions:

$$\{I_1(\tilde{\mathbf{C}}),\ I_2(\tilde{\mathbf{C}}),\ I_5(\tilde{\mathbf{C}})\}. \tag{39}$$

Let $\{\beta_1, \beta_2, \beta_3\}$ be any three invariants that are in one-to-one correspondence with the nontrivial invariants (39). Combining with J and $\bar{\lambda}$, the invariants

$$\{J,\ \bar{\lambda},\ \beta_1,\ \beta_2,\ \beta_3\} \tag{40}$$

furnish a set of basis functions.

5.2. Orthogonality between stress terms

The proposed invariant formulation enjoys the property that the ensuing stress is automatically decomposed into the pressure, the deviatoric fiber tension, and the shear terms. To show this, we start from the energy form (36). Denoted by $\boldsymbol{\sigma}$ the Cauchy stress, by virtue of the balance of mechanical power we have,

$$J\boldsymbol{\sigma}\cdot\mathbf{d} = \dot{\tilde{W}} = \frac{\partial\tilde{W}}{\partial J}J\mathbf{1}\cdot\mathbf{d} + \frac{\partial\tilde{W}}{\partial\bar{\lambda}}\bar{\lambda}\bar{\mathbf{a}}\cdot\mathbf{d} + \frac{\partial\tilde{W}}{\partial\tilde{\mathbf{C}}}\cdot\dot{\tilde{\mathbf{C}}}, \tag{41}$$

where the rate relations in (32) are utilized. Let $\tilde{\boldsymbol{\sigma}}$ be the stress component generated by $\tilde{\mathbf{C}}$, namely, $J\tilde{\boldsymbol{\sigma}}\cdot\mathbf{d} = \frac{\partial\tilde{W}}{\partial\tilde{\mathbf{C}}}\cdot\dot{\tilde{\mathbf{C}}}$, we then write

$$\boldsymbol{\sigma} = \frac{\partial\tilde{W}}{\partial J}\mathbf{1} + \frac{\bar{\lambda}}{J}\frac{\partial\tilde{W}}{\partial\bar{\lambda}}\bar{\mathbf{a}} + \tilde{\boldsymbol{\sigma}}. \tag{42}$$

Evidently,

$$\frac{\partial\tilde{W}}{\partial J}\mathbf{1} \in \mathscr{S}_1, \quad \frac{\bar{\lambda}}{J}\frac{\partial\tilde{W}}{\partial\bar{\lambda}}\bar{\mathbf{a}} \in \mathscr{S}_2. \tag{43}$$

Notice

$$\dot{\tilde{\mathbf{C}}} = \dot{\tilde{\mathbf{F}}}^{\mathrm{T}}\tilde{\mathbf{F}} + \tilde{\mathbf{F}}^{\mathrm{T}}\dot{\tilde{\mathbf{F}}} = \tilde{\mathbf{F}}^{\mathrm{T}}\left[2\tilde{\mathbf{d}}\right]\tilde{\mathbf{F}} = \tilde{\mathbf{F}}^{\mathrm{T}}[2\mathbb{P}\mathbf{d}]\tilde{\mathbf{F}},$$

where (34) is used. It is clear that

$$J\tilde{\boldsymbol{\sigma}}\cdot\mathbf{d} = \frac{\partial\tilde{W}}{\partial\tilde{\mathbf{C}}}\cdot\dot{\tilde{\mathbf{C}}} = \frac{\partial\tilde{W}}{\partial\tilde{\mathbf{C}}}\cdot\tilde{\mathbf{F}}^{\mathrm{T}}[2\mathbb{P}\mathbf{d}]\tilde{\mathbf{F}} = \mathbb{P}\left[2\tilde{\mathbf{F}}\frac{\partial\tilde{W}}{\partial\tilde{\mathbf{C}}}\tilde{\mathbf{F}}^{\mathrm{T}}\right]\cdot\mathbf{d} \quad \forall\mathbf{d}, \tag{44}$$

where the symmetry of $\mathbb{P}$ is used. Therefore

$$\tilde{\boldsymbol{\sigma}} = \frac{1}{J}\mathbb{P}\left[2\tilde{\mathbf{F}}\frac{\partial\tilde{W}}{\partial\tilde{\mathbf{C}}}\tilde{\mathbf{F}}^{\mathrm{T}}\right] \Rightarrow \tilde{\boldsymbol{\sigma}} \in \mathscr{S}_3 \oplus \mathscr{S}_4. \tag{45}$$

This proves that the stress term conjugate to $\tilde{\mathbf{C}}$ corresponds to a combination of the transverse and the in-plane shear.

It is possible to construct basis functions β_i such that the stress terms $\tilde{\boldsymbol{\sigma}}$ is further decoupled. As an example, consider the following set of invariants

$$\beta_1 = \tilde{\mathbf{C}}^2\cdot\mathbf{N}\otimes\mathbf{N}, \quad \beta_2 = \operatorname{tr}\tilde{\mathbf{C}}^{-1} - \tilde{\mathbf{C}}^{-1}\cdot\mathbf{N}\otimes\mathbf{N}, \quad \beta_3 = \operatorname{tr}\tilde{\mathbf{C}}^{-1}. \tag{46}$$

Evidently, $\beta_1 = I_5(\tilde{\mathbf{C}})$. Since $\det\tilde{\mathbf{C}} = 1$, we identify that $\beta_3 = I_2(\tilde{\mathbf{C}})$ and $\tilde{\mathbf{C}}^{-1}\cdot\mathbf{N}\otimes\mathbf{N} = K_2(\tilde{\mathbf{C}})$. It follows that $\beta_2 = K_3(\tilde{\mathbf{C}})$. These invariants therefore carry the geometrical meaning identified before, but applied to the deformation factor $\tilde{\mathbf{F}}$. We can use the Cayley–Hamilton theorem and the unity conditions (38) to obtain an alternative expression $\beta_2 = \operatorname{tr}\tilde{\mathbf{C}} - \tilde{\mathbf{C}}^2\cdot\mathbf{N}\otimes\mathbf{N}$. With the aid of (37) and the expression

$$\tilde{\mathbf{C}}^{-1} = J^{\frac{2}{3}}\bar{\lambda}^{-1}\mathbf{C}^{-1} + (1-\bar{\lambda}^{-3})\mathbf{N}\otimes\mathbf{N}, \tag{47}$$

we can write the invariants (46) explicitly as

$$\beta_1 = \frac{1}{\lambda^4}\mathbf{C}^2\cdot\mathbf{N}\otimes\mathbf{N}, \quad \beta_2 = \frac{\lambda}{J}\operatorname{tr}\mathbf{C} - \frac{1}{J\lambda}\mathbf{C}^2\cdot\mathbf{N}\otimes\mathbf{N}, \quad \beta_3 = \frac{J}{\lambda}\operatorname{tr}\mathbf{C}^{-1} - \frac{J}{\lambda^3}. \tag{48}$$

With an energy form

$$W = \bar{W}(J,\bar{\lambda},\beta_1,\beta_2,\beta_3), \tag{49}$$

the Cauchy stress function is given as

$$\boldsymbol{\sigma} = \frac{\partial \bar{W}}{\partial J}\mathbf{1} + \frac{\bar{\lambda}}{J}\frac{\partial \bar{W}}{\partial \bar{\lambda}}\bar{\mathbf{a}} + \frac{2}{J}\sum_{i=1}^{3}\frac{\partial \bar{W}}{\partial \beta_i}\mathbf{F}\frac{\mathrm{d}\beta_i}{\mathrm{d}\mathbf{C}}\mathbf{F}^{\mathrm{T}}. \tag{50}$$

A straightforward calculation yields,

$$\begin{aligned} 2\mathbf{F}\frac{\partial \beta_1}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}} &= \frac{2}{\lambda^2}[\mathbf{ba} + \mathbf{ab} - 2(\mathbf{a}\cdot\mathbf{b})\mathbf{a}],\\ 2\mathbf{F}\frac{\partial \beta_2}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}} &= \frac{\lambda}{J}[2\mathbf{b} - 2\mathbf{ba} - 2\mathbf{ab} + 2(\mathbf{b}\cdot\mathbf{a})\mathbf{a} - (\operatorname{tr}\mathbf{b} - \mathbf{b}\cdot\mathbf{a})(\mathbf{1} - \mathbf{a})]. \end{aligned} \tag{51}$$

Here, $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ is the (inverse) Finger tensor. Evidently,

$$2\mathbf{F}\frac{\partial \beta_1}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}} = \frac{2}{\lambda^2}\mathbb{P}_3[\mathbf{b}] \in \mathscr{S}_3, \quad 2\mathbf{F}\frac{\partial \beta_2}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}} = \frac{2\lambda}{J}\mathbb{P}_4[\mathbf{b}] \in \mathscr{S}_4.$$

One sees that β_1, β_2 generate a transverse shear and an in-plane shear, respectively. Consequently, the stress terms generated by J, $\bar{\lambda}$, β_1, β_2 are mutually orthogonal. The stress term by β_3 contains both in-plane and transverse shear terms. This can be verified by direct computation. Using the relations $\operatorname{tr}\mathbf{C}^{-1} = \operatorname{tr}\mathbf{b}^{-1}$, $\mathbf{b}^{-1}\cdot\mathbf{a} = \lambda^{-2}$ and $\mathbf{C}^{-1}\cdot\mathbf{N}\otimes\mathbf{N} = \lambda^2\mathbf{b}^{-2}\cdot\mathbf{a}$, taking the derivative of β_3 with respect to $\mathbf{C}$ and push-forwarding the result into Eulerian form yields,

$$2\mathbf{F}\frac{\partial \beta_3}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}} = \frac{J}{\lambda}\left[(-2\mathbf{b}^{-1} + (\operatorname{tr}\mathbf{b}^{-1} - \mathbf{b}^{-1}\cdot\mathbf{a})\mathbf{1} + (3\mathbf{b}^{-1}\cdot\mathbf{a} - \operatorname{tr}\mathbf{b}^{-1})\mathbf{a})\right]. \tag{52}$$

It can be directly verify that

$$2\mathbf{F}\frac{\partial \beta_3}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}} = -\frac{2J}{\lambda}\left[\mathbb{P}_3[\mathbf{b}^{-1}] + \mathbb{P}_4[\mathbf{b}^{-1}]\right] \in \mathscr{S}_3 \oplus \mathscr{S}_4.$$

Hence, the fifth stress term is coupled with the third and fourth stress terms.

In passing, it is noted that the invariants β_1 and β_2 are formally equivalent to the following two strain invariants deduced by Criscione et al. (2001, p. 883)

$$\bar{\beta}_3 = \log\left(\left(\frac{I_1 I_4 - I_5}{2\sqrt{I_3 I_4}}\right) + \sqrt{\left(\frac{I_1 I_4 - I_5}{2\sqrt{I_3 I_4}}\right)^2 - 1}\right), \quad \bar{\beta}_4 = \sqrt{\frac{I_5}{I_4^2} - 1}.$$

Evidently,

$$\bar{\beta}_3 = \log\left(\frac{\beta_2}{2} + \sqrt{\left(\frac{\beta_2}{2}\right)^2 - 1}\right), \quad \bar{\beta}_4 = \sqrt{\beta_1 - 1}.$$

The last invariant deduced by Criscione registers the angle between the planes of the transverse shear and the in-plane shear. It is anticipated that such an invariant can be constructed from $\tilde{\mathbf{C}}$. This line of thinking, however, is not explored here.

6. Projection formulae

Although the energy function is commonly defined in terms of invariants, from the perspective of computation, it is convenient to express the stress collectively in terms $\tilde{\mathbf{C}}$ (and J, $\bar{\lambda}$). This situation is similar to the numerical treatment of isochoric/volumetric split, where the stress contribution from the isochoric factor is conveniently computed by means of a deviatoric projection (Simo et al., 1985). A similar procedure is developed here.

Starting from the energy function (36), introduce the notations

$$\tilde{\mathbf{S}} = 2\frac{\partial \tilde{W}}{\partial \tilde{\mathbf{C}}}, \quad \tilde{\boldsymbol{\tau}} = \tilde{\mathbf{F}}\tilde{\mathbf{S}}\tilde{\mathbf{F}}^{\mathrm{T}}. \tag{53}$$

As we have already shown in (42) and (45), the Cauchy stress takes the form

$$\boldsymbol{\sigma} = \frac{\partial \tilde{W}}{\partial J}\mathbf{1} + \frac{\bar{\lambda}}{J}\frac{\partial \tilde{W}}{\partial \bar{\lambda}}\bar{\mathbf{a}} + \frac{1}{J}\mathbb{P}\tilde{\boldsymbol{\tau}}. \tag{54}$$

We are primarily interested in the stress term associated with $\tilde{\mathbf{C}}$. From (54), it is clear that the contribution can be computed using the following procedure: First, the factor $\tilde{\mathbf{F}}$ is employed to compute an auxiliary stress $\tilde{\boldsymbol{\tau}}$ in the same manner as the Kirchhoff stress is computed from the deformation gradient. Then, the stress $\tilde{\boldsymbol{\tau}}$ is projected to the appropriate stress space to yield the contribution to the Cauchy stress.

The material tangent tensors can be computed directly with the application of the chain rule. Without loss of generality, consider the case of decoupled energy function defined by

$$W = V(\tilde{\mathbf{C}}) + U(J) + K(\bar{\lambda}), \tag{55}$$

with the second Piola Kirchhoff stress given by

$$\mathbf{S} = \left[\frac{\partial \tilde{\mathbf{C}}}{\partial \mathbf{C}}\right]^{\mathrm{T}}\left(2\frac{\partial V}{\partial \tilde{\mathbf{C}}}\right) + JU'(J)\mathbf{C}^{-1} + \bar{\lambda}K'(\bar{\lambda})\left[\frac{1}{\lambda^2}\mathbf{N}\otimes\mathbf{N} - \frac{1}{3}\mathbf{C}^{-1}\right]. \tag{56}$$

The referential material tangent tensor is given by the second derivative

$$\begin{aligned}\mathbb{D} &= 4\frac{\partial^2 W}{\partial \mathbf{C}\partial \mathbf{C}}\\ &= \left[\frac{\partial \tilde{\mathbf{C}}}{\partial \mathbf{C}}\right]^{\mathrm{T}}\left(4\frac{\partial^2 V}{\partial \tilde{\mathbf{C}}\partial \tilde{\mathbf{C}}}\right)\left[\frac{\partial \tilde{\mathbf{C}}}{\partial \mathbf{C}}\right] + 4\left[\frac{\partial^2 \tilde{\mathbf{C}}}{\partial \mathbf{C}\partial \mathbf{C}}\right]^{\mathrm{T}}\left(\frac{\partial V}{\partial \tilde{\mathbf{C}}}\right) + 4K''(\bar{\lambda})\frac{\partial \bar{\lambda}}{\partial \mathbf{C}}\otimes\frac{\partial \bar{\lambda}}{\partial \mathbf{C}} + 4U''(J)\frac{\partial J}{\partial \mathbf{C}}\otimes\frac{\partial J}{\partial \mathbf{C}}\\ &\quad + 4K'(\bar{\lambda})\frac{\partial^2 \bar{\lambda}}{\partial \mathbf{C}\partial \mathbf{C}} + 4U'(J)\frac{\partial^2 J}{\partial \mathbf{C}\partial \mathbf{C}}.\end{aligned} \tag{57}$$

The contributions from $\tilde{\mathbf{C}}$ appear in the first two terms in the right-hand-side of (57). The first term relates to the second derivative of the energy function. The second term, linear in stress, arises from the nonlinearity of the tensor $\tilde{\mathbf{C}}$. The explicit expressions for the fourth order transformation tensor $\frac{\partial \tilde{\mathbf{C}}}{\partial \mathbf{C}}$ and the higher order transformation $\frac{\partial^2 \tilde{\mathbf{C}}}{\partial \mathbf{C}\partial \mathbf{C}}$ are given in Appendix A.

The Eulerian tangent tensor associated with $\tilde{\mathbf{C}}$ retains a more tractable form. Let $\mathbb{C}$ be the spatial tangent tensor related to $\mathbb{D}$ by the push-forward relation

$$\mathbb{C} = \frac{1}{J}\mathbb{F}\mathbb{D}\mathbb{F}^{\mathrm{T}} \Longleftrightarrow C^{ijkl} = \frac{1}{J}F^i_I F^j_J D^{IJKL} F^k_K F^l_L.$$

Let

$$\tilde{\mathbb{C}} = \frac{4}{J}\tilde{\mathbb{F}}\frac{\partial^2 V}{\partial \tilde{\mathbf{C}}\partial \tilde{\mathbf{C}}}\tilde{\mathbb{F}}^{\mathrm{T}} \Longleftrightarrow \tilde{C}^{ijkl} = \frac{4}{J}\tilde{F}^i_I\tilde{F}^j_J\frac{\partial^2 V}{\partial \tilde{C}_{IJ}\partial \tilde{C}_{KL}}\tilde{F}^k_K\tilde{F}^l_L \tag{58}$$

and denote $\tilde{\tau}_n = \tilde{\boldsymbol{\tau}}\cdot\mathbf{n}\otimes\mathbf{n}$. After a lengthy but straight forward computation, it is shown that the contribution from $\tilde{\mathbf{C}}$ (namely, the pushforward of the first two terms in (57)) takes the form

$$\mathbb{C} = \mathbb{P}\tilde{\mathbb{C}}\mathbb{P} + (\operatorname{tr}\tilde{\boldsymbol{\tau}} - \tilde{\tau}_n)\left[\mathbb{P}_4 + \bar{\lambda}^{-3}\mathbb{P}_3\right] - [\mathbb{P}_4(\tilde{\boldsymbol{\tau}}) \otimes (\mathbf{1} - \mathbf{a}) + (\mathbf{1} - \mathbf{a}) \otimes \mathbb{P}_4(\tilde{\boldsymbol{\tau}})]$$
$$+ 2(\bar{\lambda}^{-3} - 1)[\mathbf{a} \boxtimes \mathbb{P}_4(\tilde{\boldsymbol{\tau}}) + \mathbb{P}_4(\tilde{\boldsymbol{\tau}}) \boxtimes \mathbf{a}] - 2\bar{\lambda}^{-\frac{3}{2}}[\mathbf{a} \otimes \mathbb{P}_3(\tilde{\boldsymbol{\tau}}) + \mathbb{P}_3(\tilde{\boldsymbol{\tau}}) \otimes \mathbf{a}]. \tag{59}$$

It is clear that once $\tilde{\mathbb{C}}$ and $\tilde{\boldsymbol{\tau}}$ are obtained, the tangent tensor can be computed by means of projections and transformations. This procedure carries over the computational structure developed by Simo et al. (1985) for the treatment of the isochoric/volumetric decomposition.

7. An example model

To provide some assessments of this constitutive approach, we consider a model with additive energy function

$$W(\mathbf{C}) = k_2 \exp(c(\bar{\lambda} - 1)^2) + \frac{1}{2}k_3(\beta_1 - 1) + \frac{1}{2}k_4(\beta_2 - 2) \tag{60}$$

with β_1 and β_2 defined as in (46). We are primarily interested in assessing the predictability in modeling materials with distinct characteristics in the fiber tension and in the transverse and in-plane shear. For this reason, an exponential form is used for the fiber stretch, whereas polynomials are used for the other terms. The material is assumed to be incompressible, hence the energy function (60) determines the Cauchy stress to within a hydrostatic pressure. For numerical simulation the model is implemented in the nonlinear finite element program FEAP originally developed by Taylor (Zienkiewicz and Taylor, 1991). In the finite element simulation, a penalty term $\frac{1}{2}k_1(J-1)^2$ is added to the energy function, with k_1 setting to $10\,000k_3$. The incompressibility constraint is further treated by a mixed formulation for the volume/pressure fields and an augmented Lagrangian method, as described in Simo and Taylor (1991).

7.1. Uniaxial tension

Consider the uniaxial tension of a block in which the tensile load is applied along one of its axes. Let the motion be described by $x_i = \lambda_i X_i$, $i = 1,2,3$ where the coordinate X_1 is in the loading direction. In the first case, consider the fiber aligned in the load direction, so that X_2 and X_3 are in the isotropic plane. Due to material symmetry, we have $\lambda_2 = \lambda_3$ in this motion. A direct computation shows

$$\bar{\lambda} = \lambda_1^{\frac{2}{3}}\lambda_2^{-\frac{2}{3}}, \quad \beta_1 = 1, \quad \beta_2 = 2.$$

Notably the invariants β_1 and β_2 are constants. The incompressibility condition $\lambda_1\lambda_2\lambda_3 = 1$ further gives $\bar{\lambda} = \lambda_1$ and $\lambda_2 = \lambda_3 = \lambda_1^{-\frac{1}{2}}$. The stress function can be derived using (60), (50) and (51). Upon imposing the incompressibility condition, the stress components are given by

$$\sigma_{11} = \frac{4}{3}k_2 c \exp(c(\lambda_1 - 1)^2)\lambda_1(\lambda_1 - 1) + p,$$
$$\sigma_{22} = \sigma_{33} = -\frac{2}{3}k_2 c \exp(c(\lambda_1 - 1)^2)\lambda_1(\lambda_1 - 1) + p,$$

where p is the pressure. Using the equilibrium condition $\sigma_{22} = 0$ to eliminate the pressure we obtain the axial stress

$$\sigma_{11} = 2k_2 c \exp(c(\lambda_1 - 1)^2)\lambda_1(\lambda_1 - 1)$$

which notably depends only on the first term of the energy function.

If the load is applied at 90° to the fiber, and if we let X_2 be in the fiber direction (namely $[\mathbf{N}] = [0\,1\,0]$), we find

$$\beta_1 = 1, \quad \beta_2 = \frac{\lambda_1}{\lambda_3} + \frac{\lambda_3}{\lambda_1}.$$

In this case the stress components are found to be

$$\sigma_{11} = -\frac{1}{3}k_2 c \exp(c(\bar{\lambda}_2 - 1)^2)\bar{\lambda}_2(\bar{\lambda}_2 - 1) + \frac{k_4}{2}\left(\frac{\lambda_1}{\lambda_3} - \frac{\lambda_3}{\lambda_1}\right) + p,$$
$$\sigma_{22} = \frac{2}{3}k_2 c \exp(c(\bar{\lambda}_2 - 1)^2)\bar{\lambda}_2(\bar{\lambda}_2 - 1) + p,$$
$$\sigma_{33} = -\frac{1}{3}k_2 c \exp(c(\bar{\lambda}_2 - 1)^2)\bar{\lambda}_2(\bar{\lambda}_2 - 1) - \frac{k_4}{2}\left(\frac{\lambda_1}{\lambda_3} - \frac{\lambda_3}{\lambda_1}\right) + p,$$

where p is the pressure. Eliminating p and the exponential term by the equilibrium conditions $\sigma_{22} = \sigma_{33} = 0$, we find

$$\sigma_{11} = k_4\left(\frac{\lambda_1}{\lambda_3} - \frac{\lambda_3}{\lambda_1}\right),$$

where, given the axial stretch λ_1, the lateral stretch λ_3 together with λ_2 are determined by $\sigma_{22} = \sigma_{33} = 0$ and the incompressibility condition.

The two tension curves are depicted in Fig. 1. Finite element simulations of these two tests are also conducted. The curves clearly show an exponential behavior for the first case and a nearly linear response in the second case. In these simulation the following parameters are used:

$$k_2 = 1.35 \times 10^3 \text{ KPa}, \quad k_3 = 135 \text{ KPa}, \quad k_4 = 135 \text{ KPa}, \quad c = 1.$$

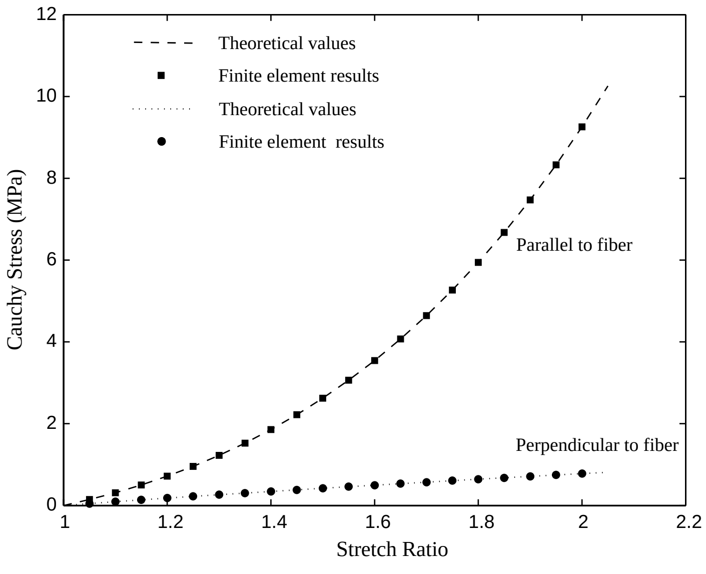

Fig. 1. Uniaxial tension: the axial stress versus the stretch.

7.2. Inextensible beam

Some peculiar deformations of transversely isotropic solids have been reported in the literature. Pipkin studied the planar deformation in materials reinforced with inextensible fibers (Pipkin, 1974, 1977). If the material is also incompressible, the motion, if all possible, may be determined by kinematic analysis alone. Particularly, if a cantilever beam with fibers parallel to the beam length is loaded transversely as shown in Fig. 2, the beam deforms by shear rather than by bending, because it can only sustain motions that are locally a pure shear along the fiber direction. The shear deformation is independent of the length of the beam and the distance along the beam where the load is applied.

This phenomenon is replicated numerically using the suggested constitutive model. A cantilever beam of length $L = 15$ in. height $H = 1$ in. is subjected to a transverse load $P = 5$ lbs, applied at $2/3L$ from the fixed end. The material is assumed nearly inextensible in the longitudinal direction with the following parameters:

$$k_2 = 10^7 \text{ Psi}, \quad k_3 = 25 \text{ Psi}, \quad k_4 = 25 \text{ Psi}, \quad c = 1.$$

The deformed configuration is depicted in Fig. 2. It is evident that the beam undergoes transverse shear motion except for in the region close to the clamped end and in the transition region near the load. The portion of beam between the load and the free end remains horizontal with zero shear stress.

Following Pipkin (1974), in the analytical solution we parameterize the local motion in terms of a pure shear in the form

$$\mathbf{F} = \mathbf{n} \otimes \mathbf{N} + (\kappa\mathbf{n} + \mathbf{m}) \otimes \mathbf{M},$$

where $\mathbf{N}$ and $\mathbf{n}$ are the tangents of the fiber line in the reference and current configurations, $\mathbf{M}$ and $\mathbf{m}$ are the corresponding normals, and κ is the amount of shear that varies with position. It follows that $\beta_1 = \kappa^2 + 1$ and $\beta_2 = 2$. Using the energy form (60) and invoking the relation $(51)_1$, the transverse shear stress is found to depend linearly on κ, the magnitude of which is given by

$$\tau = k_3\kappa.$$

In this case there is an analytical relation relating the deflection of the beam tip to the applied load (Pipkin, 1974). The force-deflection curve obtained by the finite element analysis is plotted in Fig. 3, and is found to agree with the theoretical prediction. It appears that the current model allows to sharply single out the transverse shear mode, and thus to capture the essential feature of the motion.

7.3. Torsion of hollow cylinders

Torsion of hyperelastic cylinders is a classical problem that has been extensively studied. It is known that hyperelastic cylinders do not sustain pure torsion in general, except for few material models, see Polignone

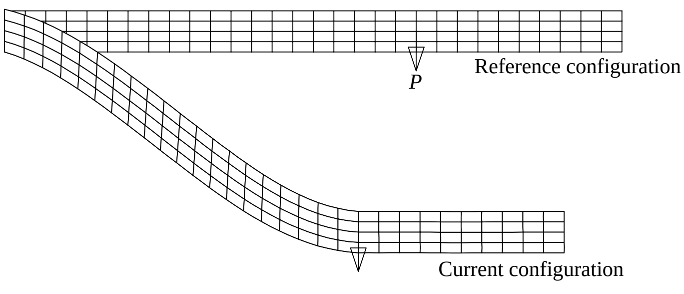

Fig. 2. Inextensible cantilever beam under transverse load: deflecting by shearing.

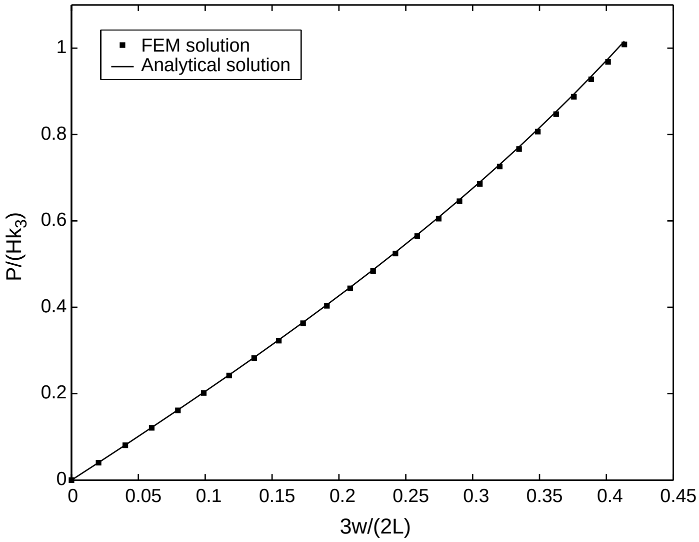

Fig. 3. Force–deflection curve.

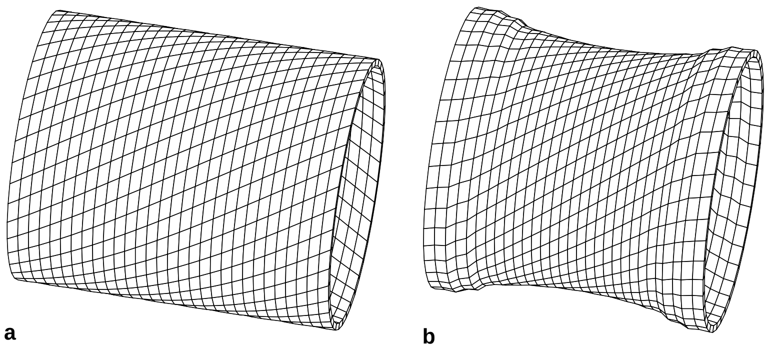

Fig. 4. Torsion of hollow cylinders with different fiber orientation: (a) fibers in the circumferential direction; (b) fibers in the longitudinal direction.

and Horgan (1991) and the references therein. For the transversely isotropic material model considered here, the torsion patterns are expected to be quite different for cylinders with different fiber orientations. In particular, pure torsion is expected when the material is inextensible in the circumferential direction. A thin-walled tube with length 5 cm, outside diameter 2 cm and wall thickness 0.1 cm is considered. The tube is clamped at one end and subjected to a torque at the free end, and is allowed to shorten in the axial direction. Material parameters are taken to be

$$k_2 = 10^6 \text{ KPa}, \quad k_3 = 135 \text{ KPa}, \quad k_4 = 135 \text{ KPa}, \quad c = 1.$$

Two different fiber-orientations are considered. In the first case, the fiber is placed along circumferential direction, while in the second case the fiber is assumed to be parallel to the cylinder length. The deformed configurations are depicted in Fig. 4. Clearly, pure torsion type of deformation is observed for the first case. In contrast, lateral contraction similar to what observed for neo-Hookean solids occurs in the second case. In addition, substantial amount of longitudinal shortening (due to fiber inextensibility) is observed in this case.

8. Concluding remarks

We discussed a constitutive formulation for transversely isotropic hyperelastic solids that may exhibit drastically different characteristics in different modes of deformation. The formulation is a logical extension of the isochoric/deviatoric decomposition used in the analysis of isotropic materials. We developed a simple multiplicative decomposition of deformation gradient, based on which a family of strain invariants that generate decoupled pressure, fiber tension and shear stresses are deduced. This framework is expected to be useful in numerical modeling of nearly incompressible and inextensible materials.

Acknowledgement

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Appendix A. Transformation tensor

In this Appendix we derive the transformation tensor $\frac{\partial \tilde{\mathbf{C}}}{\partial \mathbf{C}}$ and its derivative. Starting from (37), we have

$$\begin{aligned}\left[\partial_{\mathbf{C}}\tilde{\mathbf{C}}\right]^{\mathrm{T}}\tilde{\mathbf{S}} = {} & \frac{\lambda}{J}\tilde{\mathbf{S}} + \left[\frac{1}{2J\lambda}\mathbf{N}\otimes\mathbf{N} - \frac{\lambda}{2J}\mathbf{C}^{-1}\right](\mathbf{C}\cdot\tilde{\mathbf{S}}) + \left[\left(-\frac{2}{\lambda^2}+\frac{\lambda}{2J}\right)\mathbf{N}\otimes\mathbf{N} + \frac{\lambda^3}{2J}\mathbf{C}^{-1}\right] \\ & \times\frac{1}{\lambda^4}(\mathbf{CN}\otimes\mathbf{NC}\cdot\tilde{\mathbf{S}}) + \left(\frac{1}{\lambda^4}-\frac{1}{J\lambda}\right)\left[\tilde{\mathbf{S}}\mathbf{CN}\otimes\mathbf{N}+\mathbf{N}\otimes\mathbf{NC}\tilde{\mathbf{S}}\right]\end{aligned} \tag{61}$$

for any fixed second order tensor $\tilde{\mathbf{S}}$. Making use of the following identities

$$\begin{aligned}&\mathbf{F}\tilde{\mathbf{S}}\mathbf{F}^{\mathrm{T}} = J^{\frac{2}{3}}\bar{\lambda}^{-1}\left[\tilde{\mathbf{F}}\tilde{\mathbf{S}}\tilde{\mathbf{F}}^{\mathrm{T}} + (\bar{\lambda}^{\frac{3}{2}}-1)\mathbb{P}_4(\tilde{\mathbf{F}}\tilde{\mathbf{S}}\tilde{\mathbf{F}}^{\mathrm{T}}) + (\bar{\lambda}^3-1)(\tilde{\mathbf{F}}\tilde{\mathbf{S}}\tilde{\mathbf{F}}^{\mathrm{T}}\cdot\mathbf{a})\mathbf{a}\right],\\ &\mathbf{F}\tilde{\mathbf{S}}\mathbf{F}^{\mathrm{T}}\mathbf{a} + \mathbf{a}\mathbf{F}\tilde{\mathbf{S}}\mathbf{F}^{\mathrm{T}} = J^{\frac{2}{3}}\left[\bar{\lambda}^{\frac{1}{2}}\mathbb{P}_4(\tilde{\mathbf{F}}\tilde{\mathbf{S}}\tilde{\mathbf{F}}^{\mathrm{T}}) + 2\bar{\lambda}^2(\tilde{\mathbf{F}}\tilde{\mathbf{S}}\tilde{\mathbf{F}}^{\mathrm{T}}\cdot\mathbf{a})\mathbf{a}\right],\end{aligned} \tag{62}$$

one can deduce that

$$\mathbf{F}\left[\left[\partial_{\mathbf{C}}\tilde{\mathbf{C}}\right]^{\mathrm{T}}\tilde{\mathbf{S}}\right]\mathbf{F}^{\mathrm{T}} = \mathbb{P}\left[\tilde{\mathbf{F}}\tilde{\mathbf{S}}\tilde{\mathbf{F}}^{\mathrm{T}}\right]$$

which gives the formula (45). Moreover, taking derivative of the transformation (61) (assuming $\tilde{\mathbf{S}}$ fixed) yields,

$$\begin{aligned}\left[\partial^2_{\mathbf{CC}}\tilde{\mathbf{C}}\right]^{\mathrm{T}}\tilde{\mathbf{S}} \equiv {} & \frac{\partial}{\partial\mathbf{C}}\left[[\partial_{\mathbf{C}}\tilde{\mathbf{C}}]^{\mathrm{T}}\tilde{\mathbf{S}}\right] \\ = {} & \tilde{\mathbf{S}}\otimes\left(\frac{1}{2J\lambda}\mathbf{N}\otimes\mathbf{N}-\frac{\lambda}{2J}\mathbf{C}^{-1}\right) + \left(\frac{1}{2J\lambda}\mathbf{N}\otimes\mathbf{N}-\frac{\lambda}{2J}\mathbf{C}^{-1}\right)\otimes\tilde{\mathbf{S}} \\ & + \left(-\frac{1}{4J\lambda^3}\mathbf{C}\cdot\tilde{\mathbf{S}} + \left(\frac{6}{\lambda^8}-\frac{3}{4J\lambda^5}\right)\mathbf{CN}\otimes\mathbf{NC}\cdot\tilde{\mathbf{S}}\right)\mathbf{N}\otimes\mathbf{N}\otimes\mathbf{N}\otimes\mathbf{N} \\ & - \left(\frac{1}{4J\lambda}\mathbf{C}\cdot\tilde{\mathbf{S}} + \frac{1}{4J\lambda^3}\mathbf{CN}\otimes\mathbf{NC}\cdot\tilde{\mathbf{S}}\right)\left[\mathbf{N}\otimes\mathbf{N}\otimes\mathbf{C}^{-1}+\mathbf{C}^{-1}\otimes\mathbf{N}\otimes\mathbf{N}\right] \\ & + \left(\frac{\lambda}{4J}\mathbf{C}\cdot\tilde{\mathbf{S}} - \frac{1}{4J\lambda}\mathbf{CN}\otimes\mathbf{NC}\cdot\tilde{\mathbf{S}}\right)\left[2\mathbb{I}_{\mathbf{C}^{-1}}+\mathbf{C}^{-1}\otimes\mathbf{C}^{-1}\right]\end{aligned}$$

$$
\begin{aligned}
&+\left[\tilde{\mathbf{S}}\mathbf{C}\mathbf{N}\otimes\mathbf{N}+\mathbf{N}\otimes\mathbf{N}\mathbf{C}\tilde{\mathbf{S}}\right]\otimes\left[\left(-\frac{2}{\lambda^6}+\frac{1}{2J\lambda^3}\right)\mathbf{N}\otimes\mathbf{N}+\frac{1}{2J\lambda}\mathbf{C}^{-1}\right]\\
&+\left[\left(-\frac{2}{\lambda^6}+\frac{1}{2J\lambda^3}\right)\mathbf{N}\otimes\mathbf{N}+\frac{1}{2J\lambda}\mathbf{C}^{-1}\right]\otimes\left[\tilde{\mathbf{S}}\mathbf{C}\mathbf{N}\otimes\mathbf{N}+\mathbf{N}\otimes\mathbf{N}\mathbf{C}\tilde{\mathbf{S}}\right]\\
&+\left(\frac{1}{\lambda^4}-\frac{1}{J\lambda}\right)\left[\tilde{\mathbf{S}}\boxtimes(\mathbf{N}\otimes\mathbf{N})+(\mathbf{N}\otimes\mathbf{N})\boxtimes\tilde{\mathbf{S}}\right].
\end{aligned}
\tag{63}
$$

Here, $\mathbb{I}_{\mathbf{C}^{-1}}$ is the pull-back of the Eulerian fourth order identity tensor; in components, $[\mathbb{I}_{\mathbf{C}^{-1}}]^{IJKL}=\frac{1}{2}([C^{-1}]^{IK}[C^{-1}]^{JL}+[C^{-1}]^{IL}[C^{-1}]^{JK})$. Pushing-forward the right hand side of (63) and collecting terms yields the Eulerian form reported in (59).

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